Highly Eccentric Kozai Mechanism and GW Observation for Neutron Star Binaries

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The Kozai mechanism for a hierarchical triple system could reduce the merger time of inner eccentric binary emitting gravitational waves (GWs), and has been qualitatively explained with the secular theory that is derived by averaging short-term orbital rotations. However, with the secular theory, the minimum value of the inner pericenter distance could be artificially blocked by the back-reaction of GW emission. Compared with traditional predictions, the actual evolution of an eccentric inner binary could be accompanied by (i) a higher characteristic frequency of the pulse-like GWs around its pericenter passages, and (ii) a larger residual eccentricity at its final inspiral phase. These findings would be important for GW astronomy with the forthcoming advanced detectors.

I. INTRODUCTION

Today, large-scale laser interferometers are under development to attain a world-wide network of second-generation GW detectors [1]. Their overall sensitivities will be improved by a factor of ~ 10, with drastic noise reduction at the lower frequency regime down to ~ 10Hz [1]. Accordingly, understandings of basic properties of potential astrophysical sources have become significant, more than ever.

One of the most promising targets of these detectors is inspiral of a neutron star binary (NSB), and in this paper we focus our attention to GW observation for NSBs. From identified samples in our Galaxy, NSBs are expected to have very small residual eccentricities ($O(10^{-5})$) around 10Hz [2, 3].

Meanwhile, it has been pointed out that the Kozai mechanism might play important roles for compact binary mergers [4–6]. This mechanism works for hierarchical triple systems, and oscillates pericenter distances of inner binaries, due to exchange of angular momenta between the inner/outer orbits [8]. This characteristic feature can be qualitatively understood with the secular theory for which, following a perturbative method in analytical mechanics, we effectively average out short-term fluctuations associated with both the inner/outer orbital rotations [9, 10]. Since energy loss due to GW emission depends strongly on pericenter distance, the Kozai mechanism can largely reduce the merger time of an inner NSB of a triple system. This interesting possibility has been actively discussed mostly with the secular theory including the averaging operations [4–6] (see also [7]).

In this paper, we show that, with the secular theory, the minimum value of the inner pericenter distance could be artificially blocked by the back-reaction of GW emission, in a causally unreasonable manner. To handle evolution of a highly eccentric inner binary, we need to properly resolve the two orbital rotations without taking their averages. Using both simple analytical and numerical methods, we show that the actual evolution of an inner NSB could be largely different from that predicted by the secular theory. For the inner NSB, this could result in (i) a higher characteristic frequency of the pulse-like GWs around the pericenter passages, (ii) a higher residual eccentricity at the final inspiral phase, and (iii) a shorter merger time. All of these changes could be more than one order of magnitude. Our findings (i) and (ii) are significant for the advance detectors and their data analyses. While quantitative evaluation for the merger rate requires detailed astronomical assumptions and is beyond scope of this paper, the last one (iii) indicates a higher merger rate for NSBs of triples in star clusters [5]. This is because the outer third body would be frequently perturbed there.

In this paper, we only discuss relativistic effects for hierarchical triples, but tidal effects around planets also depend strongly on orbital distance [12]. For extra-solar planetary systems (*e.g.* Hot Jupiters [13, 14]), an investigation similar to this work would be worth considering.

II. SECULAR THEORY

We study evolution of a hierarchical triple system of point masses m_0, m_1 and m_2. We basically use the geometrical unit with $G = c = (m_0 + m_1 + m_2) = 1$. The inner binary is composed by m_0 and m_1, and we denote its semimajor axis by a_1. In the next section, we also introduce astrophysical units, considering m_0-m_1 as a NSB. For the outer third body m_2, we define its semimajor axis a_2, relative to the mass center of the inner binary (total mass $M_1 \equiv m_0 + m_1$). Likewise, we use the labels $j = 1$ and 2 for the inner and outer orbital elements (*e.g.* e_1 for the inner eccentricity), and assume hierarchical orbital configurations with $\alpha \equiv a_1/a_2 \ll 1$.

First, we briefly discuss long-term secular evolution of the triple system in Newtonian dynamics, following the approach developed by von Zeipel [9]. By suitably using canonical transformations, we effectively average the short-term fluctuations associated with both the inner and outer mean anomalies l_1 and l_2 (the instantaneous angular positions of the inner/outer point masses [12]). The relevant Hamiltonian after the averaging operations, can be evaluated perturbatively with the expansion parameter $\alpha \ll 1$. The leading order (quadrupole) term

$H'_{qd} = O(\alpha^2)$ is given by [4–6, 10]

$$H'_{qd} = C_{qd}\left[(2+3e_1^2)(1-3\theta^2) - 15e_1^2(1-\theta^2)\cos 2\omega_1\right] \tag{1}$$

with $C_{qd} \equiv m_0 m_1 m_2 \alpha^2/[16M_1 a_2(1-e_2^2)^{3/2}]$ and the argument of the inner pericenter ω_1 [12]. Here we define $\theta \equiv \cos I$ with the relative inclination angle I between the inner/outer orbits. We denote the next order (octupole) term by $H'_{oc}(= O(\alpha^3))$. For our secular analysis of the inner binary, we keep up to this term for the gravitational perturbation externally induced by m_2. But there exists a relation $H'_{oc} \propto (m_0 - m_1)$, resulting in $H'_{oc} = 0$ for $m_0 = m_1$ [10]. Later we use this property to examine possible effects of the sub-leading terms.

Next, we mention general relativistic corrections to the system, using the post-Newtonian (PN) expansion. The lowest order (1PN) term H'_{1pn} for our hierarchical configuration is obtained after averaging the inner mean anomaly l_1 as [4–6, 9]

$$H'_{1pn} = -\frac{3m_0 m_1 M_1}{a_1^2(1-e_1^2)^{1/2}}\chi_{1pn}. \tag{2}$$

Here we introduced an auxiliary parameter χ_{1pn} for later discussions ($\chi_{1pn} = 1$ unless otherwise stated). At this stage, our effective Hamiltonian H'_c for the secular evolution is given by

$$H'_c = H'_{qd} + H'_{oc} + H'_{1pn}, \tag{3}$$

and the system is conservative (thus putting the subscript "c" above) [4–6]. Using canonical equations and transformations of variables, we have *e.g.*

$$\left(\frac{d\omega_1}{dt}\right)_c = 6C_{qd}\left(\frac{4\theta^2}{G_1} + \cdots\right) + \text{O.T.} + \frac{3\chi_{1pn}}{a_1(1-e_1^2)}\left(\frac{M_1}{a_1}\right)^{3/2}, \tag{4}$$

$$\left(\frac{de_1}{dt}\right)_c = 30C_{qd}\frac{e_1(1-e_1^2)}{G_1}(1-\theta^2)\sin 2\omega_1 + \text{O.T.}, \tag{5}$$

and $(da_1/dt)_c = 0$. Here we defined $G_1 = m_0 m_1\left[a_1(1-e_1^2)/M_1\right]^{1/2}$ and put O.T. for terms originating from H'_{oc} [4, 6].

The triple system becomes dissipative at the 2.5PN order, due to emission of GWs. Given our hierarchical configuration, the dissipation predominantly works for the inner binary, and we include its effects only for a_1 and e_1, using standard formulae for isolated eccentric binaries [11]. Combining these with the conservative contributions, we can write down the final expressions for the secular evolution such as $d\omega_1/dt = (d\omega_1/dt)_c$,

$$\frac{da_1}{dt} = -\frac{64m_0 m_1 M_1}{5a_1^3(1-e_1^2)^{7/2}}\left(1 + \frac{73}{24}e_1^2 + \frac{37}{96}e_1^4\right), \tag{6}$$

$$\frac{de_1}{dt} = -\frac{304m_0 m_1 M_1 e_1}{15a_1^4(1-e_1^2)^{5/2}}\left(1 + \frac{121}{304}e_1^2\right) + \left(\frac{de_1}{dt}\right)_c, \tag{7}$$

with strong dependencies on $1 - e_1$, and $da_2/dt = 0$ [4, 6]. These secular equations have been widely used for analyzing long-term evolutions of relativistic hierarchical triple systems.

III. NUMERICAL RESULTS

In this section, we numerically discuss the Kozai mechanism for relativistic hierarchical triples, first using the secular equations and then directly integrating the PN equations for three-body systems. While a triple system has many parameters, we fix most of them to concisely explain our new findings.

In our geometrical unit, we fix the masses at $M_1 = 0.2$, $m_2 = 0.8$, and the initial orbital parameters at $a_1 = 3.57\times10^5$, $a_2 = 60a_1 = 2.14\times10^7 \equiv a_{2i}$ (*i.e.* initially $\alpha = 1/60$), $e_1 = 0.2$ and $e_2 = 0.6$. We also set the initial angular variables at $\omega_1 = \pi/2$ and $\Omega_1 = \omega_2 = 0$ (Ω_1: the longitude of the inner ascending node [12]). For the secular theory, the remaining parameter is the initial inclination I_i. We explore the regime $I_i \sim 90°$ for which an inner binary can merge in a short time (also preferable for costly direct calculations).

For actual astrophysical system, we presume that the inner binary is a NSB with their total mass $M_1 = 2.8M_\odot$. Then the initial axes correspond to $a_1 = 0.05$ AU and $a_2 = a_{2,i} \equiv 3$AU. Below, instead of the direct time variable t, we use the effective outer rotation cycles $N_2 \equiv t/P_{2i}$ defined with the initial orbital period $P_{2i} = 2\pi a_{2i}^{-3/2}$ (corresponding to 1.38yr). The primary GW frequency of a quasi-circular inner binary becomes 10Hz (∼lower end of the advanced detectors) at $a_1 = a_{1cr} \equiv 34.6$.

Since observed NSBs have nearly equal masses (with relative difference of $\lesssim 7\%$ [2]), we mainly set $m_1 = m_2 = 0.1$ in the geometrical unit. For an isolated binary with a semimajor axis $a = 0.05$AU and masses $m_0 = m_1 = 1.4M_\odot$, the merger time due to GW emission becomes 1.0×10^{10}yr even for $e = 0.7$.

As mentioned earlier, the octupole term H'_{oc} vanishes for $m_0 = m_1$. In order to safely estimate its potential effects, we subsidiary examine an excessive case $(m_0, m_1) = (0.11, 0.09)$.

A. Results with the Secular Theory

As an example for predictions of the secular theory, in Fig.1, we provide the inner orbital elements a_1 and $r_{p1} \equiv a_1(1-e_1)$ (pericenter distance), as functions of the outer cycles N_2. Their ratio r_{p1}/a_1 is identical to $(1-e_1)$. The basic parameters for this calculation are given in the caption.

The inner binary merges at $N_{2m} = 1209$ that is considerably smaller than the cycles $N_{2m} = O(10^{10-11})$ for isolated binaries with moderate initial eccentricities [5, 6]. Due to the Kozai mechanism, the inner eccentricity e_1

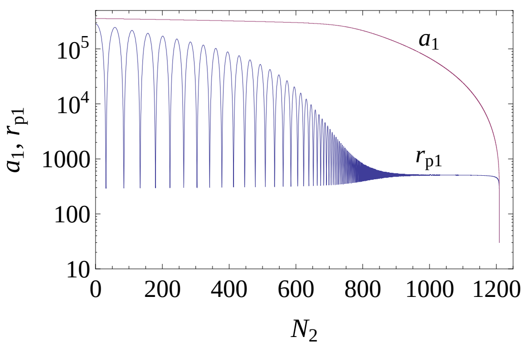

FIG. 1: Evolution of the inner semimajor axis a_1 and pericenter distance $r_{p1} = (1 - e_1)a_1$. These results are obtained with the traditional secular theory. We set $m_1 = m_2 = 0.1$, $m_2 = 0.8$ with initial inclination $I_i = 91°$, and initial eccentricities $e_1 = 0.2$ and $e_2 = 0.6$. The inner binary merges at the outer cycles $N_{2m} = 1209$.

oscillates with the minimum distances of $r_{p1} \simeq 300$. The energy of the inner binary is radiated mostly around these epochs. As discussed in the literature [5, 6], the oscillation amplitude of e_1 decreases gradually due to the 1PN apsidal precession (the last term in Eq.(4)), and, at $N_2 \gtrsim 1000$, the inner elements evolve, as if an isolated binary. The binary becomes nearly circular at the final phase close to the merger. The two elements $e_1(\ll 1)$ and a_1 there are related to their earlier values a_{1e} and $e_{1e}(\simeq 1)$ as $e_1 \simeq (425/304)^{145/242}\,(a_1/2a_{1e}(1-e_{1e}))^{19/12}$ [11]. At the critical separation $a_1 = a_{1cr}$, the residual eccentricity becomes $e_{1cr} = 5.3 \times 10^{-3}$.

In Fig.2, using the symbols on the solid lines, we show the duration N_{2m} and the residual eccentricity e_{1cr} at $a_1 = a_{1cr}$ for $I_i \sim 90°$. The results (circles) for $(m_0, m_1) = (0.1, 0.1)$ are similar to those (triangles) for $(m_0, m_1) = (0.11, 0.09)$. Therefore, for the present parameters, the octupole term plays a minor role, and the perturbative expansion itself is effective for the secular theory (see also [14, 15]).

B. Direct Three-Body Calculations

Now we move to direct three-body calculations. We use the PN equations of motions for spinless three-body systems [16], and handle the three particles equivalently. In addition to the Newtonian and 1PN terms, we included the dissipative 2.5PN terms, but excluded the time consuming 2PN terms (briefly discussed later).

For numerical integration, we apply a fourth-order Runge-Kutta scheme with an adaptive time-step control [17]. We terminate our runs, when the inner semimajor axis decreases to $a_1 = a_{1cr}$ or when the separation between m_0/m_1 becomes less than $10M_1$. The later condition reflects our perturbative (PN) treatment of nonlinear gravity, but no run encountered this condition. For numerical evaluation of a_j and e_j $(j = 1, 2)$, we use the con-

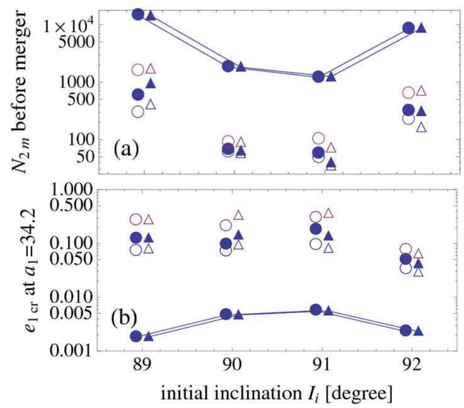

FIG. 2: (a) The circles $(m_0 = m_1 = 0.1)$ and triangles $(m_0 = 0.11, m_1 = 0.09)$ represent the outer rotation cycles N_{2m} before the mergers of the inner binaries (slightly displaced horizontally to prevent overlaps of symbols). The symbols with lines are obtained from the traditional secular theory. These without lines are from direct three-body calculations. For each inclination I_i, totally fifty runs with random initial mean anomalies are analyzed, and we show the median values (filled symbols), first (25%) and third (75%) quantiles (open symbols). (b); The residual eccentricity e_{1cr} of the inner binaries at the semimajor axes $a_1 = a_{1cr} = 34.6$ (corresponding to the primary GW frequency of 10Hz for a NSB).

secutive maximum $((1+e_j)a_j)$ and minimum $((1-e_j)a_j)$ of the two orbits.

For the direct calculations, we need to specify the initial mean anomalies l_j. Since three-body problem depends strongly on initial conditions, we randomly distribute the initial mean anomalies to examine statistical trends of evolutions. For each initial inclination I_i and mass combination in Fig.2, we made 50 runs, and evaluated their median values and first/third quantiles of the durations N_{2m} and the residual eccentricities e_{1cr}. For $m_0 = m_1 = 0.1$ and $I_i = 90°$, we additionally made 50 runs, including the 2PN terms, and obtained the median values $N_m = 78.3$ and $e_{1cr} = 0.136$ that are close to the corresponding ones in Fig.2. Therefore, for our analyses, the 2PN effect would not be important.

We found that, in the direct calculations, the outer parameters a_2 and e_2 stay nearly at their initial values, in agreement with the secular theory. However, Fig.2 shows that the duration N_{2m} and residual e_{1cr} are totally different.

To closely look at these discrepancies, we show an illustrative sample among the 50 runs for $I_i = 91°$ and $m_0 = m_1 = 0.1$. This run ended at $N_{2m} = 52.5$ with the residual $e_{1cr} = 0.313$ (close to the upper quantile in Fig.2). If we simply use the outer cycle N_2 (as in Fig.1), the semimajor axis a_1 comes to appear merely as a step function, and we cannot resolve its rapid final evolution. Therefore, for Fig.3, we use the logarithm of the remain-

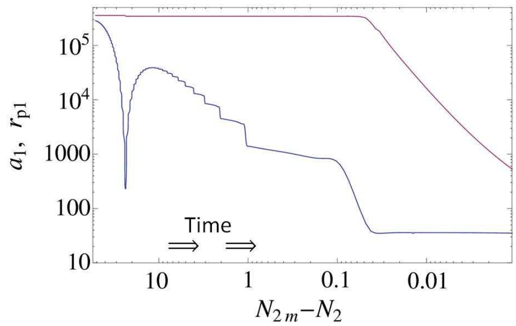

FIG. 3: Similar to Fig.1, but given from a direct three-body calculation. The inner binary takes $e_{1cr} = 0.313$ at $a_1 = a_{1cr}$, and merges at the outer cycle $N_{2m} = 52.5$. For the horizontal axis, we use the remaining outer cycle $\Delta N \equiv N_{2m} - N_2$ before the inner merger.

ing cycles $\Delta N \equiv N_{2m} - N_2$ before the merger.

We can see that, up to $\Delta N = O(0.1)$, the axis a_1 is nearly a constant, but the pericenter distance r_{p1} has modulation of period ~ 1. This reflects the outer eccentric motion characterized by l_2, and does not exist in the secular theory for which we average out the outer body, roughly speaking, as a continuous ring.

At the turning point $\Delta N \sim 4\times10^{-2}$, the quantity $1-e_1$ takes a minimum value, corresponding to $r_{p1} = 38$, and the axis a_1 starts to decrease rapidly from $a_1 = 3.0\times10^5$. The distance r_{p1} stays nearly at a constant for a while. The inner binary evolves almost independently on the outer body m_2, and emits pulse-like GWs around the pericenter passages. This waveform has a characteristic frequency $(M_1/r_{p1}^3)^{1/2}/\pi \sim 10$Hz that is ~ 30 times higher than the counterpart in Fig.1. This change would be encouraging for ground-based GW observation.

IV. DISCUSSIONS

Actually, the large reduction of r_{p1} sheds light on an important issue of the traditional secular theory for highly eccentric orbits. Here we discuss this from the viewpoint of GW emissions. For an orbit with $1 - e_1 \ll 1$, GW emission is dominated at the pericenter passages, and the radiated energy there is given as $\delta E \sim -85\pi(m_0m_1)^2M_1^{1/2}/(12\sqrt{2}r_{p1}^{7/2})$, depending strongly on r_{p1} [18]. Therefore, in order to lose a fraction Y of the total energy $-m_0m_1/2a_1$, the close approach r_{p1} should be

$$r_{p1,gw} = 33\left(\frac{a_1}{3.0\times10^5}\right)^{2/7}\left(\frac{Y}{0.2}\right)^{-2/7} \quad (8)$$

with our model mass parameters, a fiducial value $Y = 0.2$, and a_1 at the turning point.

Meanwhile, for the secular theory, we can estimate the local minimum of $1 - e_1$ from Eq.(7) [6]. Neglecting the octupole terms, we obtain the minimum pericenter distance $r_{p1,min} = a_1(1 - e_1)_{min}$ as

$$r_{p1,min} \simeq 60\left(\frac{a_2/a_1}{71}\right)\left(\frac{a_1}{3.0\times10^5}\right)^{1/6}\left(\frac{X}{1}\right)^{-1/3} \quad (9)$$

with the factor $X \equiv \sin^2 I|\sin 2\omega_1| \le 1$.

Therefore, even with the conservative setting $X = 1$, the minimum distance $r_{p1,min}$ of the secular theory is much larger than Eq.(8), and only a small fraction $Y \ll 1$ of energy can be lost during the single passage.

Now we carefully look at Eq.(7). The fluctuations of the inner angular momentum ($\propto \sqrt{a_1(1-e_1^2)}$) due to the outer body is mostly induced when the inner separation is $O(a_1)$ (considering the torque from the tidal force) and the radiation reaction does not work virtually. In other words, aside from probability, no physical mechanism prohibits a nearly radial orbit ($1 - e_1 \ll 1$) in the separation regime. However, Eq.(7) indicates that the minimum value of $1 - e_1$ is blocked by the radiation reaction, and the underlying analysis falls into causally unreasonable structure.

To properly handle the evolution, we need to sufficiently resolve the inner orbital motion without making its average. Here the concrete position of the outer body would be also important to estimate the fluctuations induced to the small inner angular momentum [19]. If orbital elements (*e.g.* $1 - e_1$) evolve more slowly than the two orbital periods, the traditional double averages might be efficient, but we need careful studies for highly eccentric binaries that could be critical to GW astronomy.

Fig.2 shows that the residual eccentricity e_{1cr} could be much larger than the estimation by the traditional secular theory. This is closely related to the difference of the pericenter distances. For a quasi-circular binary, the residual eccentricity could be probed through the associated phase modulation of GWs [3]. For a NSB detectable with advanced detectors at SNR~ 15, the resolution of the residual value e_{1cr} (at 10Hz) would be $\Delta e_{1cr} \simeq 0.01$ [3]. Interestingly, this is just between the two predictions in Fig.2. While this figure is given for specific sets of parameters, our results would be intriguing for the upcoming GW detectors.

Acknowledgments

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